MLC School

2016
YEAR 12 TRIAL HSC EXAMINATION

Mathematics Extension 1

Name: .... Teacher: ....

Date:	2016
Weighting:	40 %

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write in blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in questions 11 - 14
- A separate reference sheet is provided

Total Marks – **70**

Section 1 Pages 3 - 5

10 marks

Attempt Questions 1 - 10

Allow about 15 minutes for this section

Section 2 Pages 6 - 10

60 marks

Attempt Questions 11 - 14

Allow about 1 hour 45 minutes for this section

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Section I

10 marks
Attempt Questions 1-10
Allow 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

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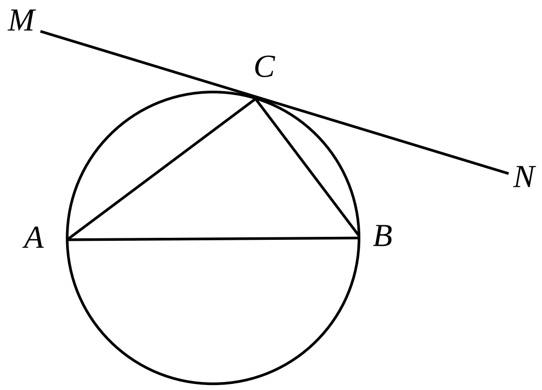
1 In the diagram, AB is a diameter of the circle and MCN is the tangent to the circle at C. $\angle CAB = 35°$. What is the size of $\angle MCA$?



(A) 35°

(B) 45°

(C) 55°

(D) 65°

2 Which of the following is the domain of $y = \cos^{-1}\left(\frac{x}{2}\right)$?

(A) $-2 \le x \le 2$

(B) $0 < x < 2\pi$

(C) $-\frac{1}{2} \le x \le \frac{1}{2}$

(D) $0 \le x \le \frac{\pi}{2}$

3 The solution to the inequality $\dfrac{x-1}{x+2} > 0$ is:

(A) $-2 < x < 1$

(B) $-1 < x < 2$

(C) $x < -2$ or $x > 1$

(D) $x < -1$ or $x > 2$

4 The acute angle between the lines $2x - y = 0$ and $kx - y = 0$ is equal to $\dfrac{\pi}{4}$.
What is the value of k?

(A) $k = -3$ or $k = -\dfrac{1}{3}$

(B) $k = -3$ or $k = \dfrac{1}{3}$

(C) $k = 3$ or $k = -\dfrac{1}{3}$

(D) $k = 3$ or $k = \dfrac{1}{3}$

5 After t years the number N of individuals in a population is given by $N = 400 + 100e^{-0.1t}$.
What is the difference between the initial population size and the limiting population size?

(A) 100

(B) 300

(C) 400

(D) 500

6 The point dividing the interval from $A(-3,1)$ to $B(1,-1)$ externally in the ratio 3:1 is:

(A) $\left(0, -\tfrac{1}{2}\right)$

(B) $\left(-1, \tfrac{1}{2}\right)$

(C) $(-5, 2)$

(D) $(3, -2)$

7 The expression $\dfrac{\sin 3\theta}{\sin\theta}-\dfrac{\cos 3\theta}{\cos\theta}$ can be simplified to

(A) $\sin 2\theta - \cos 2\theta$

(B) $\sin 2\theta + \cos 2\theta$

(C) $\tan 2\theta$

(D) 2

8 Which of the following is an asymptote of the curve $y=\dfrac{x^2-4}{x}$?

(A) $y=x$

(B) $x=2$

(C) $x=1$

(D) $y=0$

9 Which of the following is an expression for $1+\sec x$ in terms of t given $t=\tan\dfrac{x}{2}$?

(A) $\dfrac{2}{1+t^2}$

(B) $\dfrac{2}{1-t^2}$

(C) $\dfrac{2t}{1+t^2}$

(D) $\dfrac{2t}{1-t^2}$

10 Which of the following is a solution of the equation $2^x=5$?

(A) $x=\sqrt{5}$

(B) $x=\log_e x$

(C) $x=\dfrac{\log_e 5}{\log_e 2}$

(D) $x=\dfrac{\log_e 2}{\log_e 5}$

Section II

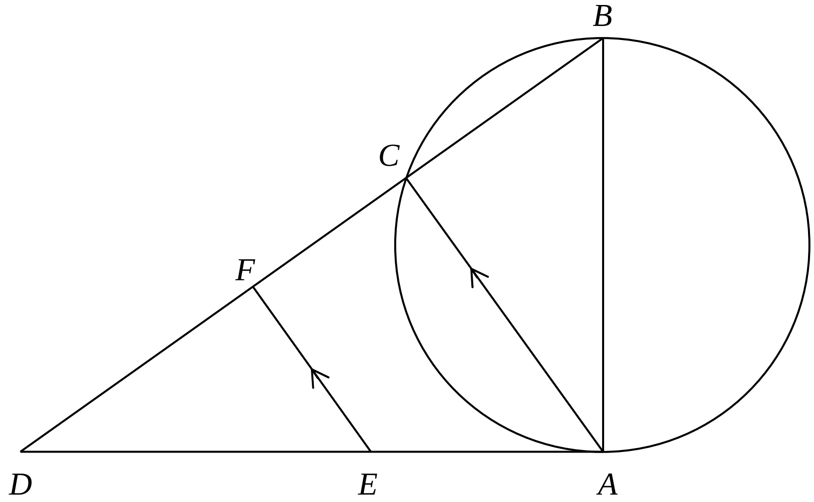
60 marks
Attempt Questions 11-14
Allow 1 hour 45 minutes for this section

Answer in separate writing booklets for this section. Start each question in a new booklet.

Question 11. (15 marks) **Marks**

(a) Find $\lim_{x \to 0} \frac{\sin 3x}{x}$. **2**

(b)



AB is a diameter of the circle and C is a point on the circle. The tangent to the circle at A meets BC produced at D. E is a point on AD and F is a point on CD such that EF is parallel to AC.

(i) Give a reason why $\angle EAC = \angle ABC$. **1**

(ii) Hence or otherwise show that $EABF$ is a cyclic quadrilateral. **2**

(iii) Explain why BE is a diameter of the circle through E, A, B and F. **1**

(Question 11 continues on the next page)

(Question 11 continued) **Marks**

(c) Solve for *x*: $\dfrac{2x-3}{x} \le 4$ **2**

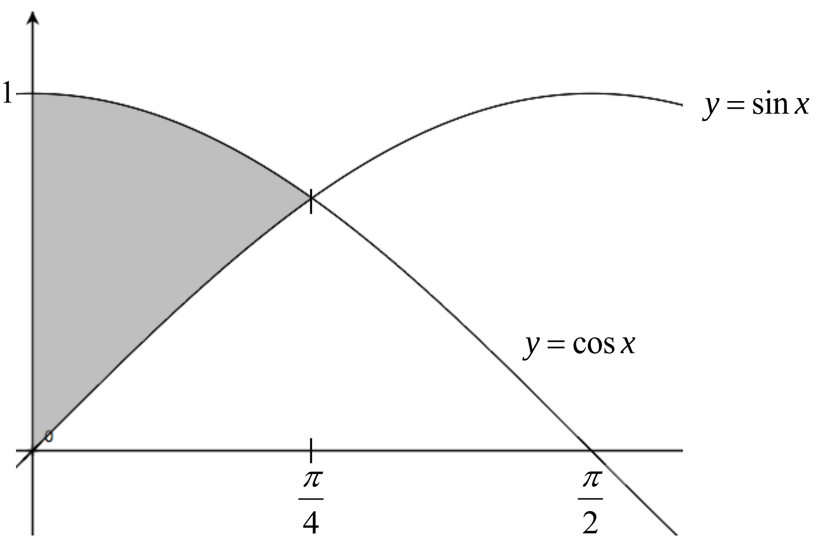
(d) Use Mathematical induction to show that for all positive integers $n \ge 2$,

$$2\times 1+3\times 2+4\times 3+...+n(n-1)=\frac{n(n^2-1)}{3}$$ **4**

(e) Find the exact value of $\displaystyle\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$ **2**

Question 12. **(15 marks)** (Start a new booklet) **Marks**

(a)



The region bounded by the curves $y=\sin x$ and $y=\cos x$ between $x = 0$ and $x=\dfrac{\pi}{4}$ is rotated through one complete revolution around the *x*-axis. Find the volume of the solid of revolution. **3**

(b) (i) Show that $\dfrac{d}{dx}\left(\sin^2 x\right)=\sin 2x$ **1**

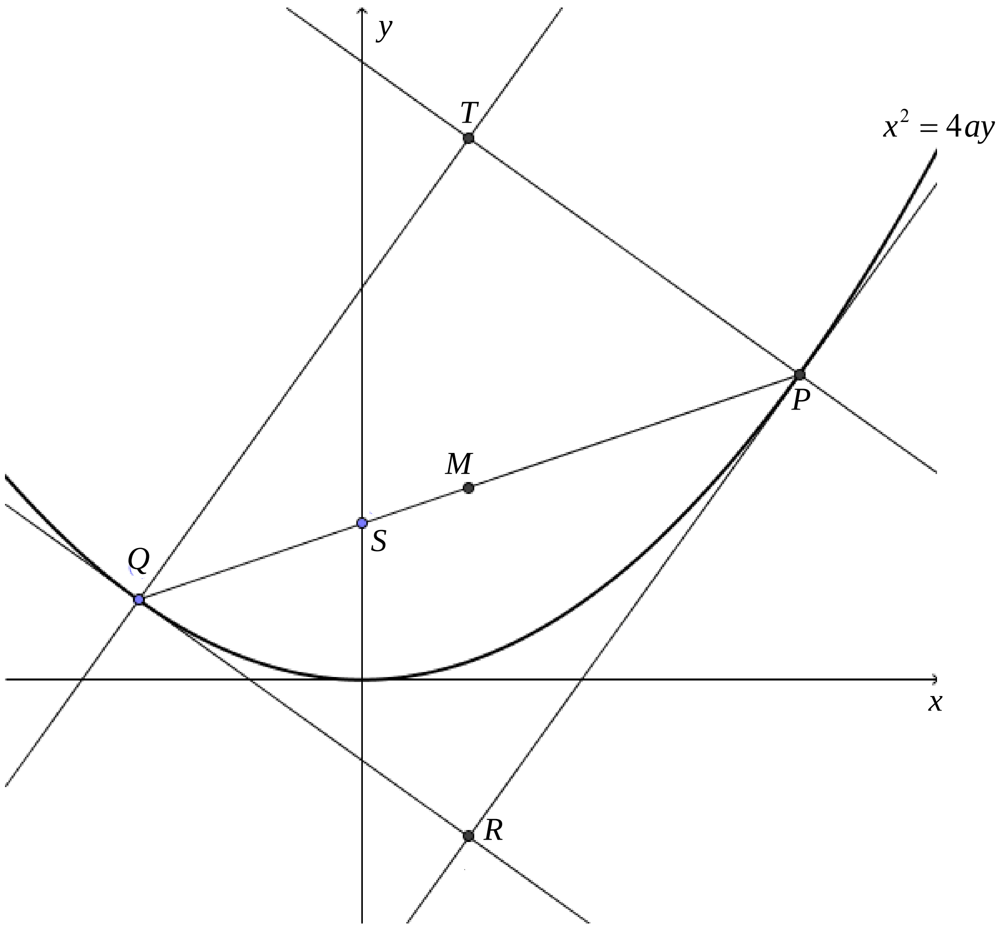
(ii) Hence use the substitution $u=\sin^2 x$ to evaluate $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\sin 2x}{1+\sin^2 x}\,dx$ **3**

(Question 12 continues on the next page)

(Question 12 continued) **Marks**

(c) Find $\int \frac{1+2x}{1+x^2}dx$. **3**

(d)



The points $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ lie on the parabola $x^2 = 4ay$. The tangents to the parabola at *P* and *Q* intersect at *R*. The normals at *P* and *Q* intersect at *T*. The point *M* is the midpoint of the chord *PQ*. The point *S* is the focus (0, *a*).

(i) Find the coordinates of *M*. **1**

(ii) Show that $pq = -1$ if *PQ* is a focal chord. **2**

(iii) By considering the *x*-coordinates of the three points or otherwise, show that if *PQ* is a focal chord, then the points *R*, *T* and *M* are collinear. **3**

Question 13. (15 marks) (Start a new booklet) **Marks**

(a) Consider the function $f(x)=(x+2)^2-9,\ -2\le x\le 2$.

(i) Find the equation of the inverse function $f^{-1}(x)$. **1**

(ii) On the same diagram, sketch the graphs of $y=f(x)$ and $y=f^{-1}(x)$, showing clearly the coordinates of the endpoints and the intercepts on the coordinate axes. **3**

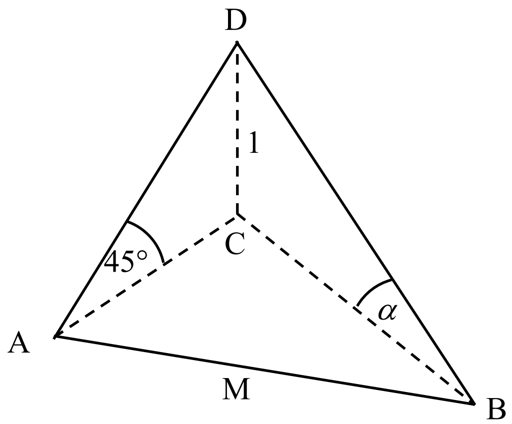
(iii) Find the *x*-coordinate of the point of intersection of the curves $y=f(x)$ and $y=f^{-1}(x)$. **2**

(b) Consider the function $f(x)=\tan^{-1}(x-1)$.

(i) Sketch the curve $y=f(x)$, showing clearly the equations of any asymptotes and the intercepts on the coordinate axes. **2**

(ii) Find the equation of the tangent to the curve $y=f(x)$ at the point where $x = 1$. **2**

(c)



CD is a vertical pole of height 1 metre that stands with its base C on horizontal ground. A is a point due South of C such that the angle of elevation of D from A is 45°. B is a point due East of C such that the angle of elevation of D from B is α. M is the midpoint of AB.

(i) Show that $BC=\cot\alpha$ and hence show that $AB=\text{cosec}\ \alpha$. **3**

(ii) Show that $CM=\dfrac{1}{2}\text{cosec}\ \alpha$ **2**

Question 14. (15 marks) (Start a new booklet) **Marks**

(a) A particle is performing Simple Harmonic Motion in a straight line. At time t seconds, it has displacement x metres from a fixed point O in the line, velocity v ms^{-1} given by $v = -12\sin\left(2t + \frac{\pi}{3}\right)$ and acceleration $\ddot{x}$ ms^{-2}. Initially the particle is 5 metres to the right of O.

(i) Find an expression for x. **1**

(ii) Show that $\ddot{x} = -4(x-2)$. **2**

(ii) Find the extremes of motion. **2**

(iii) Find the time taken by the particle to return to its starting point for the first time. **2**

(b) After t hours, the number of individuals in a population is given by $N = 500 - 400e^{-0.1t}$.

(i) Sketch the graph of N as a function of t, showing clearly the initial population size and the limiting population size. **2**

(ii) Show that $\frac{dN}{dt} = 0.1(500 - N)$. **1**

(iii) Find the population size for which the rate of growth of the population is half the initial rate of growth. **1**

(c) A particle is moving in a straight line. After time t seconds, it has displacement x metres from a fixed point O in the line, velocity v ms^{-1} given by $v = \sqrt{x}$, and acceleration a ms^{-2}. Initially the particle is 1 metre to the right of O.

(i) Show that a is constant. **1**

(ii) Express x in terms of t. **2**

(iii) Find the distance travelled by the particle during the third second of motion. **1**

END OF EXAM

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SOLUTIONS

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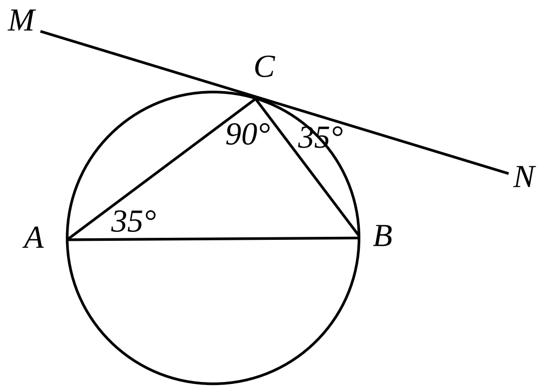

(A) 35°

(B) 45°

(C) 55° ☜

(D) 65°

2 Which of the following is the domain of $y = \cos^{-1}\left(\frac{x}{2}\right)$?

(A) $-2 \le x \le 2$ ☜

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(C) $\tan 2\theta$

(D) 2 ☜

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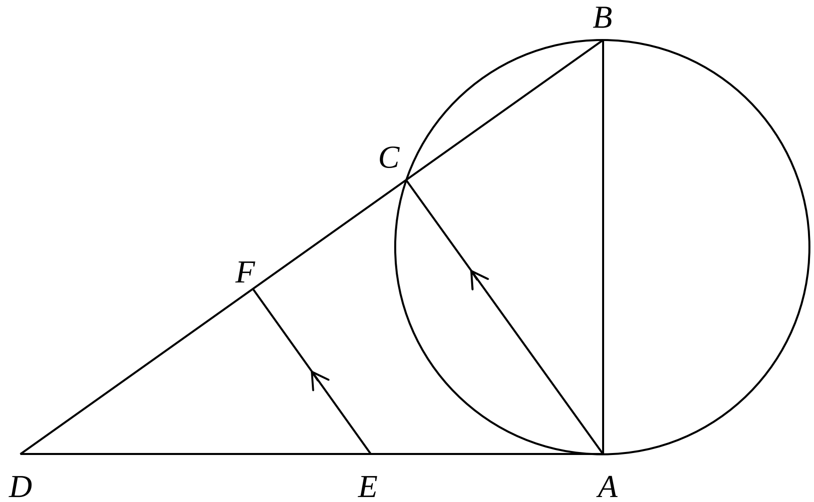
Question 11. (15 marks) **Marks**

(a) Find $\lim_{x \to 0} \frac{\sin 3x}{x}$. **2**

$\lim_{x \to 0} \frac{\sin 3x}{x} = 3\lim_{x \to 0} \frac{\sin 3x}{3x} = 3$

(b)



AB is a diameter of the circle and *C* is a point on the circle. The tangent to the circle at *A* meets *BC* produced at *D*. *E* is a point on *AD* and *F* is a point on *CD* such that *EF* is parallel to *AC*.

(i) Give a reason why $\angle EAC = \angle ABC$. **1**

Angle between tangent and chord equals angle in alternate segment

(ii) Hence or otherwise show that *EABF* is a cyclic quadrilateral. **2**

$\angle DEF = \angle EAC$ (corresponding angles in $\parallel$ lines)

$\angle DEF = \angle ABC$

$\therefore$ *EABF* is cyclic (exterior angle equals opposite interior angle)

Marks

(iii) Explain why BE is a diameter of the circle through E, A, B and F. **1**

$\angle EAB = 90°$ (tangent perpendicular to radius)

$\therefore$ BE is a diameter of circle $EABF$ (angle in semi-circle = 90°)

(c) Solve for x: $\dfrac{2x-3}{x} \le 4$ **2**

$$x(2x-3) \le 4x^2$$
$$x(2x-3) - 4x^2 \le 0$$
$$x(2x-3-4x) \le 0$$
$$-x(3+2x) \le 0$$
$$x > 0 \quad \text{or} \quad x \le -\frac{3}{2}$$

(d) Use Mathematical induction to show that for all positive integers $n \ge 2$,

$$2\times1 + 3\times2 + 4\times3 + ... + n(n-1) = \frac{n(n^2-1)}{3}$$ **4**

Let $n = 2$: $2\times1 = \dfrac{2(2^2-1)}{3} = 2$ $\therefore$ true when $n = 2$

Assume true for $n = k$: $2\times1 + 3\times2 + 4\times3 + ... + k(k-1) = \dfrac{k(k^2-1)}{3}$

Let $n = k + 1$ and show that

$$2\times1 + 3\times2 + 4\times3 + ... + k(k-1) + (k+1)k = \frac{(k+1)\left((k+1)^2-1\right)}{3}$$

$$\begin{aligned}
LHS &= 2\times1 + 3\times2 + 4\times3 + ... + k(k-1) + (k+1)k \\
&= \frac{k(k^2-1)}{3} + k(k+1) \\
&= \frac{k\left((k^2-1) + 3(k+1)\right)}{3} \\
&= \frac{k(k^2+3k+2)}{3} \\
&= \frac{k(k+2)(k+1)}{3} \\
&= RHS
\end{aligned}$$

$\therefore$ true for $n = k + 1$ if true for $n = k$

$\therefore$ true for all positive integers $n \ge 2$

(e) Find the exact value of $\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$ **2**

$$\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\frac{x}{2}\right]_{\sqrt{2}}^{\sqrt{3}}$$

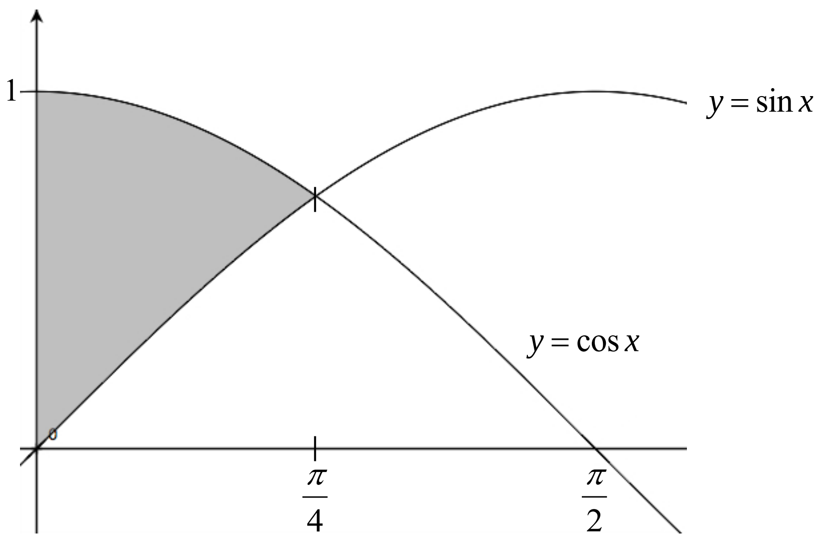
$$= \left(\sin^{-1}\frac{\sqrt{3}}{2}\right) - \left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$$

$$= \frac{\pi}{3} - \frac{\pi}{2}$$

$$= \frac{\pi}{6}$$

Question 12. (15 marks) (Start a new booklet) **Marks**

(a)



The region bounded by the curves $y = \sin x$ and $y = \cos x$ between $x = 0$ and $x = \frac{\pi}{4}$ is rotated through one complete revolution around the x-axis. Find the volume of the solid of revolution. **3**

$$V = \pi\int_0^{\pi/4} \cos^2 x - \sin^2 x\,dx$$

$$= \pi\int_0^{\pi/4} \cos 2x\,dx$$

$$= \frac{\pi}{2}\left[\sin 2x\right]_0^{\pi/4}$$

$$= \frac{\pi}{2}\left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$= \frac{\pi}{2}$$

(b) (i) Show that $\frac{d}{dx}\left(\sin^2 x\right) = \sin 2x$ **1**

$$\frac{d}{dx}\left(\sin^2 x\right) = 2\sin\cos x = \sin 2x$$

(ii) Hence use the substitution $u = \sin^2 x$ to evaluate $\int_{\pi/4}^{\pi/3} \frac{\sin 2x}{1+\sin^2 x}\,dx$ **3**

$$u = \sin^2 x \Rightarrow du = 2\sin x\cos x dx = \sin 2x dx$$

$$x = \frac{\pi}{3} \Rightarrow u = \frac{3}{4}$$

$$x = \frac{\pi}{4} \Rightarrow u = \frac{1}{2}$$

$$\int_{\pi/4}^{\pi/3} \frac{\sin 2x}{1+\sin^2 x}dx = \int_{1/2}^{3/4} \frac{1}{1+u}du$$

$$= \Big[\ln(1+u)\Big]_{1/2}^{3/4}$$

$$= \ln\frac{7}{4} - \ln\frac{3}{2}$$

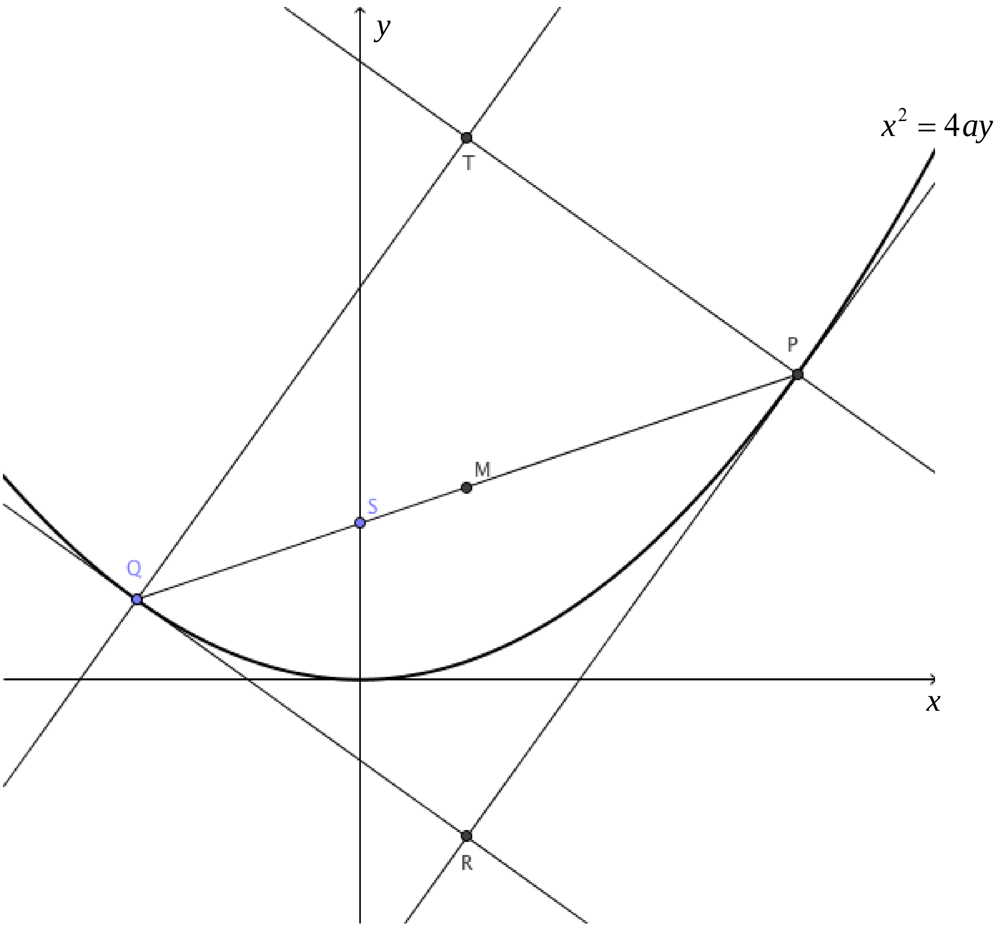
$$= \ln\frac{7}{6}$$

(c) Find $\int \frac{1+2x}{1+x^2}\,dx$. **3**

$$\int \frac{1+2x}{1+x^2}dx = \int \frac{1}{1+x^2}dx + \int \frac{2x}{1+x^2}dx$$

$$= \tan^{-1} x + \ln\left(1+x^2\right) + c$$

(d)



The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. The tangents to the parabola at *P* and *Q* intersect at *R*. The normals at *P* and *Q* intersect at *T*. The point *M* is the midpoint of the chord *PQ*. The point *S* is the focus (0, *a*).

(i) Find the coordinates of *M*. **1**

M is at $\left(\dfrac{2ap+2aq}{2}, \dfrac{ap^2+aq^2}{2}\right) = \left(a(p+q), \dfrac{1}{2}a(p^2+q^2)\right)$

(ii) Show that *pq* = -1 if *PQ* is a focal chord. **2**

Equation of PQ: $y - ap^2 = \dfrac{p+q}{2}(x - 2ap)$

Substitute the coordinates of *S* into the equation:

$$a - ap^2 = \frac{p+q}{2}(0 - 2ap)$$
$$a - ap^2 = -ap^2 - apq$$
$$a = -apq$$
$$pq = -1$$

(iii) By considering the x-coordinates of the three points or otherwise, show that if PQ is a focal chord, then the points R, T and M are collinear. **3**

THE LONG WAY:

If PQ is a focal chord, then the tangents meet on the directrix $y = -a$.

R lies on the tangent at P: $y = px - ap^2$. Let $y = -a$:

$$\begin{aligned}
-a &= px - ap^2 \\
px &= ap^2 - a \\
&= a\left(p^2 - 1\right) \\
x &= \frac{a}{p}\left(p^2 - 1\right) \\
&= \frac{a}{p}\left(p^2 + pq\right) \\
&= a\left(p + q\right)
\end{aligned}$$

So R has the same x-coordinate as M.

Find the equations of the normals:

At P, $x + py = ap^3 + 2ap$

Similarly, at Q: $x + qy = aq^3 + 2aq$

Solving these simultaneously to find T:

$$\begin{aligned}
py - qy - ap^3 + aq^3 &= 2ap - 2aq \\
\left(p - q\right)y &= 2a\left(p - q\right) + a\left(p^3 - q^3\right) \\
&= 2a\left(p - q\right) + a\left(p - q\right)\left(p^2 + pq + q^2\right) \\
y &= 2a + a\left(p^2 + q^2 - 1\right) \\
x &= 2ap + ap^3 - py \\
&= 2ap + ap^3 - 2ap - ap^3 - apq^2 + ap \\
&= ap - apq^2 \\
&= ap + aq \\
&= a\left(p + q\right)
\end{aligned}$$

So T has the same x-coordinate, and thus R, M and T all lie on the same vertical line.

THE SHORT WAY

$\angle$QRP, $\angle$RPT and $\angle$RQT are all right angles (tangents $\perp$ normals)

$\therefore$ The quadrilateral RPTQ is a rectangle (all angles of quadrilateral are right angles)

$\therefore$ The midpoints of diagonals RT and PQ are the same (diagonals of a rectangle bisect each other)

$\therefore$ The points R, M and T must be collinear (midpoint M must lie on diagonal RT)

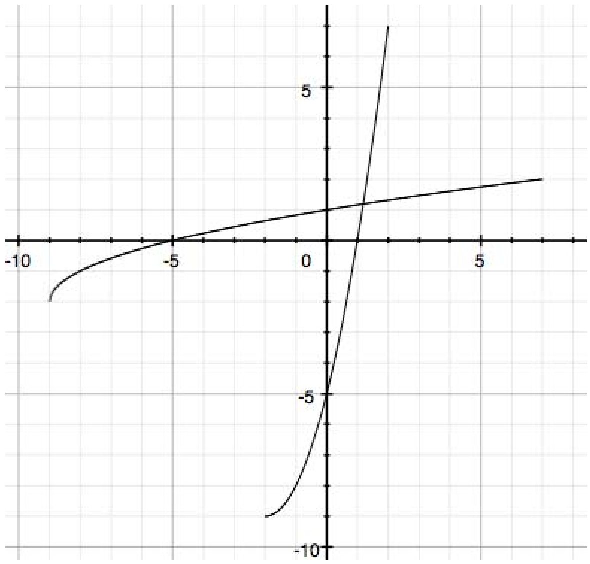
Question 13. (15 marks) (Start a new booklet) **Marks**

(a) Consider the function $f(x)=(x+2)^2-9,\ -2 \le x \le 2$.

(i) Find the equation of the inverse function $f^{-1}(x)$. **1**

$$\rightarrow x=(y+2)^2-9$$
$$x+9=(y+2)^2$$
$$\sqrt{x+9}=y+2$$
$$y=-2+\sqrt{x+9}$$

(ii) On the same diagram, sketch the graphs of $y=f(x)$ and $y=f^{-1}(x)$, showing clearly the coordinates of the endpoints and the intercepts on the coordinate axes. **3**



(iii) Find the *x*-coordinate of the point of intersection of the curves $y=f(x)$ and $y=f^{-1}(x)$. **2**

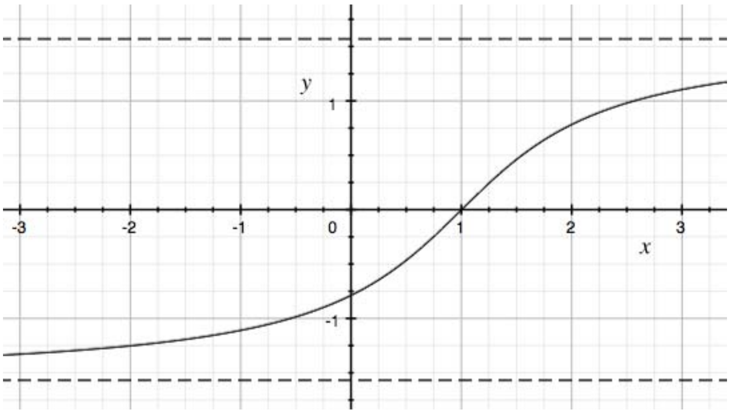
Find the intersection of $y=f(x)$ and $y=x$:

$$x=(x+2)^2-9$$
$$x=x^2+4x+4-9$$
$$x^2+3x-5=0$$
$$x=\frac{-3\pm\sqrt{3^2-4\cdot1\cdot(-5)}}{2\cdot1}=\frac{\sqrt{29}-3}{2}$$

The lines intersect at $\left(\frac{\sqrt{29}-3}{2},\frac{\sqrt{29}-3}{2}\right)$

(b) Consider the function $f(x)=\tan^{-1}(x-1)$.

(i) Sketch the curve $y = f(x)$, showing clearly the equations of any asymptotes and the intercepts on the coordinate axes. **2**



(ii) Find the equation of the tangent to the curve $y = f(x)$ at the point where $x = 1$. **2**

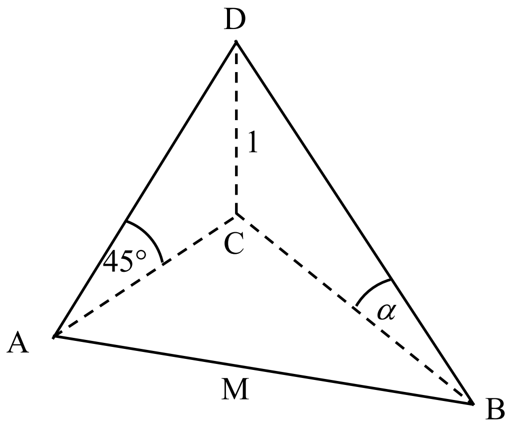
$$f(x) = \tan^{-1}(x-1)$$

$$f'(x) = \frac{1}{1+(x-1)^2} = \frac{1}{x^2-2x+2}$$

$$f'(1) = \frac{1}{1^2-2(1)+2} = 1$$

Tangent: $y - 0 = 1(x-1) \;\Rightarrow\; y = x - 1$

(c)



CD is a vertical pole of height 1 metre that stands with its base C on horizontal ground. A is a point due South of C such that the angle of elevation of D from A is 45°. B is a point due East of C such that the angle of elevation of D from B is α. M is the midpoint of AB.

(i) Show that $BC = \cot\alpha$ and hence show that $AB = \text{cosec } \alpha$. **3**

In ΔBCD,

$$\frac{CD}{BC} = \tan\alpha$$

$$\frac{BC}{1} = \frac{1}{\tan\alpha}$$

$$BC = \cot\alpha$$

In ΔABC,

$$AB^2 = AC^2 + BC^2$$

$$= 1^2 + \cot^2\alpha$$

$$= \csc^2\alpha$$

$$AB = \csc\alpha$$

(ii) Show that $CM = \frac{1}{2}\text{cosec } \alpha$ **2**

$\angle ACB = 90°$ $\therefore AB$ is the diameter of a circle with M as its centre and A, B and C are equally distant from M

$\therefore CM = AM = \frac{1}{2}AB = \frac{1}{2}\csc\alpha$

Question 14. (15 marks) (Start a new booklet) **Marks**

(a) A particle is performing Simple Harmonic Motion in a straight line. At time t seconds, it has displacement x metres from a fixed point O in the line, velocity v ms^{-1} given by $v = -12\sin\left(2t + \frac{\pi}{3}\right)$ and acceleration $\ddot{x}$ ms^{-2}. Initially the particle is 5 metres to the right of O.

(i) Find an expression for x. **1**

$$x = 6\cos\left(2t + \frac{\pi}{3}\right) + c$$

When $t = 0$, $x = 5$

$$5 = 6\cos\frac{\pi}{3} + c \quad \Rightarrow \quad c = 2$$

$$x = 6\cos\left(2t + \frac{\pi}{3}\right) + 2$$

(ii) Show that $\ddot{x} = -4(x-2)$. **2**

$$\ddot{x} = -24\cos\left(2t + \frac{\pi}{3}\right)$$

$$= -4 \times 6\cos\left(2t + \frac{\pi}{3}\right)$$

$$= -4(x-2)$$

(iii) Find the extremes of motion. **2**

Let $v = 0$:

$$-12\sin\left(2t+\frac{\pi}{3}\right)=0$$

$$2t+\frac{\pi}{3}=0,\ \pi,\ 2\pi,\ldots$$

$$x=6\cos\left(2t+\frac{\pi}{3}\right)+2$$

$$=6\cos(0)+2,\ 6\cos(\pi)+2$$

$$=8,\ -4$$

(iv) Find the time taken by the particle to return to its starting point for the first time. **2**

Let $x = 5$:

$$6\cos\left(2t+\frac{\pi}{3}\right)+2=5$$

$$6\cos\left(2t+\frac{\pi}{3}\right)=3$$

$$\cos\left(2t+\frac{\pi}{3}\right)=\frac{1}{2}$$

$$2t+\frac{\pi}{3}=\frac{\pi}{3},\frac{5\pi}{3},\ldots$$

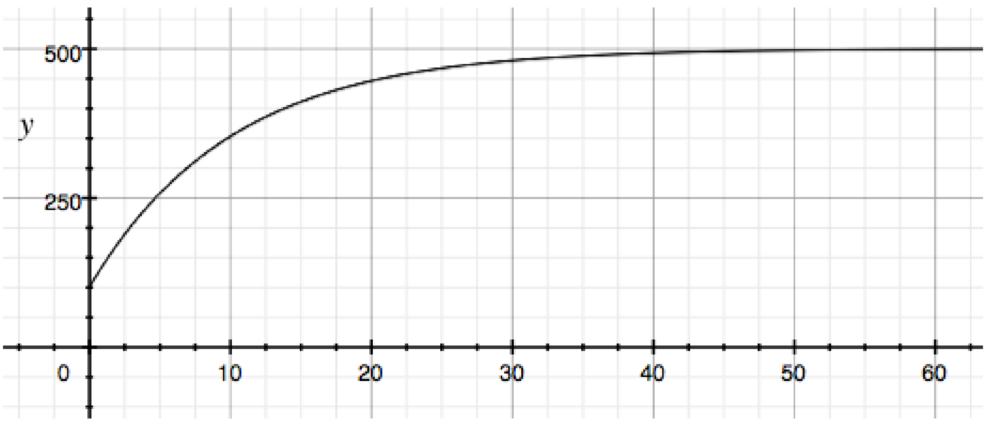
$$2t=0,\frac{4\pi}{3}$$

$$t=0,\frac{2\pi}{3}$$

The particle takes $\frac{2\pi}{3}$ seconds to return to its starting point for the first time

(b) After t hours, the number of individuals in a population is given by $N=500-400e^{-0.1t}$.

(i) Sketch the graph of N as a function of t, showing clearly the initial population size and the limiting population size. **2**



(ii) Show that $\frac{dN}{dt} = 0.1(500 - N)$. **1**

$$N = 500 - 400e^{-0.1t}$$
$$\frac{dN}{dt} = -400e^{-0.1t} \times -0.1$$
$$= 0.1\left(400e^{-0.1t}\right)$$
$$= 0.1(500 - N)$$

(iii) Find the population size for which the rate of growth of the population is half the initial rate of growth. **1**

Initial rate of growth $\frac{dN}{dt} = 0.1(500 - 100) = 40$

Let $\frac{dN}{dt} = 20$:

$$0.1(500 - P) = 20$$
$$500 - P = 200$$
$$P = 300$$

(c) A particle is moving in a straight line. After time t seconds, it has displacement x metres from a fixed point O in the line, velocity v ms^{-1} given by $v = \sqrt{x}$, and acceleration a ms^{-2}. Initially the particle is 1 metre to the right of O.

(i) Show that a is constant. **1**

$$\ddot{x} = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right)$$
$$= \frac{d}{dx}\left(\tfrac{1}{2}x\right)$$
$$= \frac{1}{2}$$

(ii) Express x in terms of t. **2**

$$\frac{dx}{dt} = x^{1/2}$$
$$\frac{dt}{dx} = \frac{1}{x^{1/2}} = x^{-1/2}$$
$$t = 2x^{1/2} + c$$

When $t = 0$, $x = 1$: $c = -2$

$$t = 2x^{1/2} - 2$$
$$\frac{1}{2}(t + 2) = x^{1/2}$$
$$x = \frac{1}{4}(t + 2)^2$$

(iii) Find the distance travelled by the particle during the third second of motion. **1**

When $t = 2$, $x = 4$. When $t = 3$, $x = 6.25$

The particle does not change direction in this time (v is always positive)

The particle travels 2.25 m during the third second.

END OF EXAM